Final Exam at 12 MTH 213, Fall 2018

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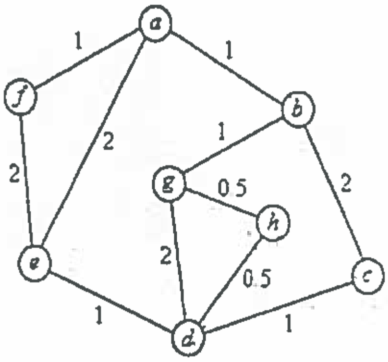
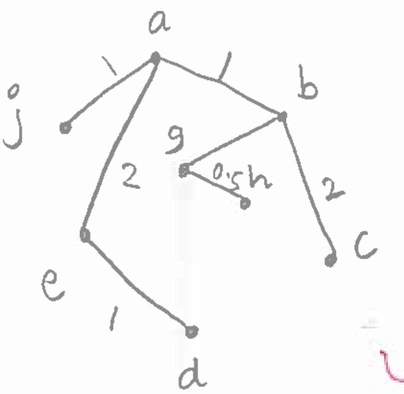
Score = $\frac{76}{80}$

QUESTION 1. (8 points)

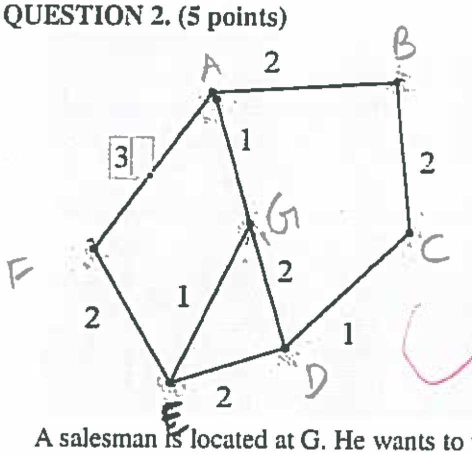
Use Dijkstra's method to find the minimum spanning tree of the below graph (you may start from vertex a).



	a	b	c	d	e	j	g	h
a	$\boxed{0}$	1a	∞	∞	2a	1a	∞	∞
b		$\boxed{1a}$	3b	∞	2a	1a	2b	∞
j			3b	∞	2a	$\boxed{1a}$	2b	∞
e			3b	3e	$\boxed{2a}$		2b	∞
g			3b	3e			$\boxed{2b}$	2.5g
h			3b	3e				$\boxed{2.5g}$
c			$\boxed{3b}$	3e				
d				$\boxed{3e}$				



QUESTION 2. (5 points)



A salesman is located at G. He wants to visit each block (each vertex) exactly once and then return to G.

1) Find all possible Hamiltonian cycle.

G - E - F - A - B - C - D - G = 13

G - D - C - B - A - F - E - G = 13

2) Find the Hamiltonian cycle with minimum weight.

Both GEFABCDG and GDCBAFEG have min weight = 13

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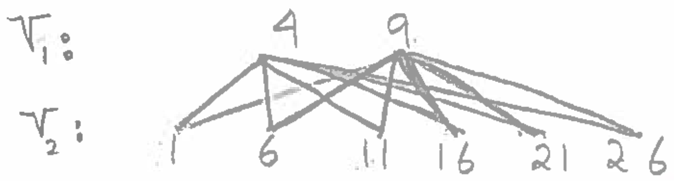
QUESTION 3. (6 points) Let $V = \{1, 4, 6, 9, 11, 16, 21, 26\}$. Two vertices in V, say a, b, are connected by an edge if and only if $a + b = 5c$ for some $c \in N^*$.

a) Draw such graph.

1 6 11 16 21 26

4 9



b) Is the graph a complete bipartite graph? if it is a $K_{n,m}$, then find n and m.

Yes, as every ~~edge~~ point in V_1 is connected by an edge to a point in V_2.

$n = 2, \; m = 6$

c) Find the diameter of the graph.

diam (G) = 2

d) Is the graph an Eulerian? If yes, then find such Eulerian circuit.

Yes. 4 - 1 - 9 - 16 - 4 - 21 - 9 - 26 - 4 is an Eulerian circuit.

e) Is the graph Hamiltonian? If yes, then find such Hamiltonian cycle.

No.

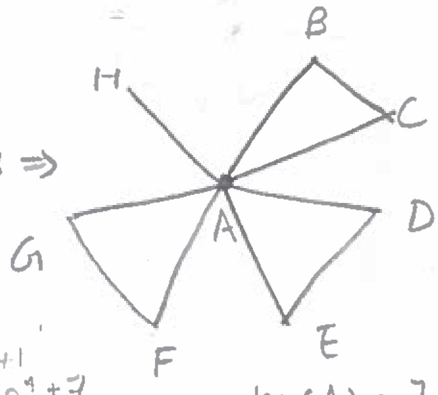
QUESTION 4. (4 points) Is the sequence 7, 2, 2, 2, 2, 2, 2, 1 graphical (i.e., is there a graph so that the vertices have the given degrees)? If yes draw such graph.

A B C D E F G H

7 | 2 2 2 2 2 2 1

⇒ | 1 1 1 1 1 1 0

⇒ | 0 1 1 1 1 0 ⇒ Σ deg = 4, We can form a graph as ⇒



deg(A) = 7

deg(B) = deg(C) = deg(D) = deg(E) = deg(F) = deg(G) = 2

deg(H) = 1

QUESTION 5. (6 points)) Consider the following code

For $k = 2$ to $(n^4 + 5)$
$S = k^4 + 3 * k + 4$ 6
For $i = 2$ to $(k + 3)$
$L = i^3 + 7 * i + 2$ 5
next i
next k

$n^4 + 5 - 2 + 1 = n^4 + 4$

$6 + 5(4) = 26$

$n^4 + 8 - 2 + 1$

$n^4 + 5 \quad 6 + 5(n^4 + 8 - 2) \cdot 6 + 5(n^4 + 7) = 6 - \quad n^4 + 7$

$6 + 35 + 5n^4 = 41 + 5n^4$

(i) Find the exact number of addition, subtraction, multiplication that the code executed.

no of operations in outer loop = 6

no of operations in inner loop = 5

no of iterations of outer loop = $n^4 + 5 - 2 + 1 = n^4 + 4$

for trial 1 (K = 2): total operations = $6 + 5(5 - 2 + 1) = 6 + 5(4) = 26$

⋮

for trial $n^2 + 4$ (K = $n^4 + 5$) total operations = $6 + 5(n^4 + 8 - 2 + 1) = 6 + 5(n^4 + 7) = 41 + 5n^4$

∴ Total number of operations in code = $\dfrac{67 + 5n^4 (n^4 + 4)}{2}$

$= \dfrac{5n^8 + 67n^4 + 20n^4 + 268}{2}$

(ii) Find the complexity of the code.

$O(S_n) = n^8$

QUESTION 6. (4 points) $A = \{4, 6, 7, 8, 9, 11, 13, 15\}$ and let $B = P(A)$ (i.e., B is the power set of A).
(a)Find $|B|$.

$|B| = 2^{|A|} = 2^8 = 256$.

(b) Define " $=$ " on B such that $\forall c, d \in B$, $c" = "d$ if and only if $c \cap \bar{d} = \emptyset$ (note $\bar{d}$ means $A - d$). By example, convince me that " $=$ " is not transitive and hence " $=$ " is not an equivalence relation on B.

$\phi \cap A' = \phi$, $\phi \cap \{4\}' = \phi$

$A \cap \{4\}' \neq \phi$

$\therefore$ $\phi " = " A$, $\phi " = " \{4\}$ but $A " \neq " \{4\}$. Therefore not transitive

(c) Let $F = \{c \in B \mid |C| = 4\}$. Find $|F|$ (note that $|c|$ means the cardinality of C) .

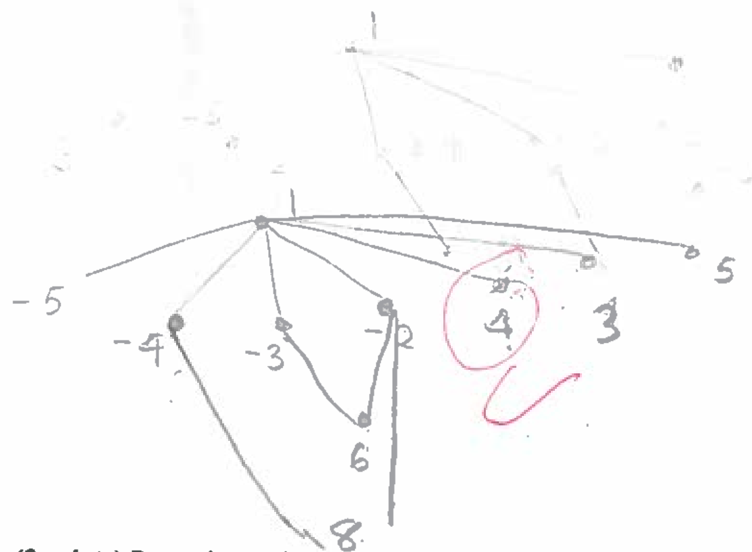
$|F| = {}^{256}C_4$

ϕ, A

$\phi = A$

QUESTION 7. Let $A = \{-5, -4, -3, -2, 1, 3, 4, 5, 6, 8\}$. Define " $\leq$ " on A such that $\forall\, a, b \in A$, "$a \leq b$" if and only if $a = bc$ for some $c \in A$. Then " $\leq$ " is a partial order relation on A (Do not show that).

(i) **(4 points)** Draw the Hassee diagram of such relation



$-5 \leq -5$, $-5 \leq 1$

$-4 \leq -4$, $-4 \leq 1$,

$-3 \leq -3$, $-3 \leq 1$,

$-2 \leq -2$, $-2 \leq 1$,

$1 \leq 1$

$3 \leq 3$, $3 \leq 1$

$4 \leq 4$, $4 \leq 1$

$5 \leq 5$, $5 \leq 1$

$6 \leq 6$, $6 \leq 1$, $6 \leq -3$, $6 \leq -2$

$8 \leq 8$, $8 \leq 1$, $8 \leq -2$, $8 \leq -4$

$-3 = (-2) \times 3$

$3 = 1 \times 3$

$4 = -2 \times -2$

(ii) **(3 points)** By staring at the Hassee diagram, if possible, find

a. $8 \vee 6$ -2

b. $-4 \wedge 8$ 8

c. $-2 \wedge 3$ dne

d. $4 \wedge -4$ dne

e. Is there a $c \in A$ such that $a \leq c$ for every $a \in A$? If yes, find c $c = 1$

f. Is there an $m \in A$ such that $m \leq a$ for every $a \in A$? If yes, find m dne

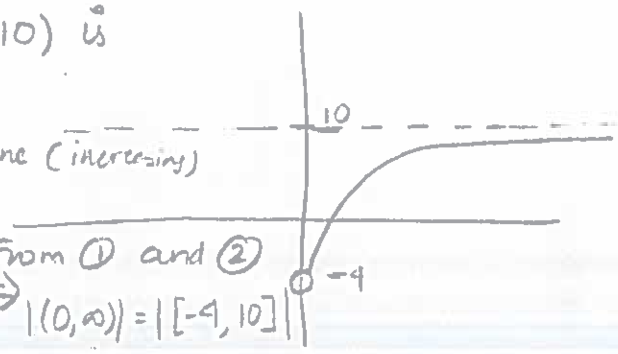
QUESTION 8. (4 points) Convince me that $|(0, \infty)| = |[-4, 10]|$ (you need to use the concept of bijective function).

A bijective func from $(0, \infty) \longrightarrow (-4, 10)$ is

$f(x) = -14e^{-x} + 10$

The $f(x)$ is onto (codomain = range) and one-one (increasing)

$\therefore$ $|(0, \infty)| = |(-4, 10)|$ —— ①



Now, $|(-4, 10)| \cup \{-4, 10\}| = |(-4, 10)|$ ②

From ① and ② $\Rightarrow$ $|(0, \infty)| = |[-4, 10]|$

QUESTION 9. (4 points)

(i) How many 6-digit odd integers STRICTLY greater than 400003 can be formed using the digits $\{2, 3, 4, 5, 6, 7, 8\}$ such that the fourth digit must be an even integer.

$5 \times 7 \times 7 \times 7 \times 4 \times 3 = 20580$

(ii) There are 849 balls and there are 10 holes (very deep holes). The holes are labeled A, A, A, A, A B, B, B, C, C. 507 balls must be placed in A-holes (i.e., maybe all of them in one A-hole, or in two A-holes or in three A-holes or in four A-holes or in five A-holes), 33 balls must be placed in B-holes (see my earlier comment), and the remaining balls must be placed in C-Holes (again, see my earlier comment). Then there are at least n balls that are placed in the same hole (such hole could be an A-hole, or a B-hole, or a C-hole). What is the maximum value of n?

A: $n_A = \left\lceil \frac{507}{5} \right\rceil = 102$ | B: $n_B = \left\lceil \frac{33}{3} \right\rceil = 11$ | C: $n_C = \left\lceil \frac{309}{2} \right\rceil = 155$

$\boxed{n_{max} = 155}$

QUESTION 10. (4 points)
Given $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Let f be a bijective function from S onto S such that

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 5 & 1 & 2 & 8 & 4 & 3 \end{pmatrix}$$

(i) Find f^2.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 8 & 2 & 7 & 6 & 3 & 1 & 5 \end{pmatrix}$$

(ii) Find the least positive integer n such that $f^n = I$, where I is the identity map (i.e., $I(a) = a$ for every $a \in S$)

least cycles : $(1\ 7\ 4)(2\ 6\ 8\ 3\ 5)$

$LCM(3, 5) = 15 \quad \therefore f^{15} = I$

QUESTION 11. (6 points) Let $A = \{-5, -4, -3, -2, -1, 1, 2, 3, 4, 5, 6, 7, 8\}$. Define "=" on A such that $\forall a, b \in A$, $a" = "b$ if and only if $a \pmod 3) = b \pmod 3)$. Then "=" is an equivalence relation. Do not show that.

(i) Find all equivalence classes of A.

$[-5] = \{-2, 1, 4, 7, -5\}$

$[-4] = \{-1, 2, 5, 8, -4\}$

$[-3] = \{3, 6, -3\}$

$3 - 2 = 1$

$-5 \bmod 3 = 3 - 2 = 1$

$-4 \bmod 3$

$3 - 1 = 2$

$-2 \bmod 3$

$= 3 - 2 \bmod 3$

$= 1$

$-1 \bmod 3$

$-1 \bmod 3$

$3 - 1 = 2$

$-1 \bmod 3$

$-4 \bmod 3$

$3 \cdot 2$

$3 - 1$

$2 \bmod 3$

$-3 \bmod 3 =$

$8 \bmod 3$

$1 \bmod 3$

4m

(ii) view "=" as a subset of $A \times A$. How many elements does "=" have?

$|=| = 5^2 + 5^2 + 3^2 = 59$

QUESTION 12. (5 points) Let $m = gcd(28, 128)$. Then find a, b such that $m = 28a + 128b$

$$28 \overset{4}{\overline{)128}} \quad \begin{array}{r}112\\ \hline 16\end{array} \rightarrow 16 \overset{1}{\overline{)28}} \quad \begin{array}{r}16\\ \hline 12\end{array} \rightarrow 12 \overset{1}{\overline{)16}} \quad \begin{array}{r}12\\ \hline 4\end{array} \rightarrow 4 \overset{3}{\overline{)12}} \quad \begin{array}{r}12\\ \hline 0\end{array}$$

$m = gcd(28, 128) = 4$

$$\begin{aligned} 4 &= 16 - 12 \\ &= 128 - 28(4) - (28 - 16) \\ &= 128 - 28(5) + 128 - 28(4) \\ &= 128(2) + 28(-9) \end{aligned}$$

256

$a = -9$
$b = 2$

QUESTION 13. (6 points) Use math induction and convince me that $15 \mid (3^{(8m+3)} - 12)$ for every $m \geq 1$.

1) for $m = 1$: $3^{8+3} - 12 = 177135$

$15 \mid 177135 \Rightarrow 15 \mid 3^{11} - 12 \Rightarrow 15 \mid 3^{8m+3} - 12$ for $m = 1$

2) Let's say the claim is true for some $m > 1$.

i.e, $3^{8m+3} - 12 = 15k$ (where $k \in \mathbb{Z}$) — 2

3) We have to prove the claim for $m+1$.

$$\begin{aligned} 3^{8(m+1)+3} - 12 &= 3^{8m+3} \cdot 3^8 - 12 = 3^{8m+3} \cdot 3^8 - 12 \cdot 3^8 + 12 \cdot 3^8 - 12 \\ &= 3^8(3^{8m+3} - 12) + 12(3^8 - 1) \\ &= \underbrace{3^8(15k)}_{\text{from } 2} + \underbrace{12 \times 6560}_{\text{divisible by } 15} \end{aligned}$$

$\therefore 15 \mid 3^{8(m+1)+3} - 12$ is true.

$\Rightarrow 15 \mid 3^{8m+3} - 12$ for $\forall m \geq 1$.

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QUESTION 14. (6 points) Let X be number of students in MTH 111. Given $0 < X < 90$ such that $X \ (mod\ 9) = 2$ and $4X \ (mod\ 10) = 6$. Use the Chinese remainder Theorem (CRT) and find all possible values of X.

$x \ (\text{mod } 9) = 2$ $\quad c_1 \ n_1 \ m_1$ $\quad 2 \ 9 \ 10$ $\quad \gcd(9,10) = 1 \Rightarrow$ CRT applies

$4x \ (\text{mod } 10) = 6 \quad 6 \ 10 \ 9$

$10x = 1 \ (\text{mod } 9)$

$x_1 = 1$

$4x = 6 \ (\text{mod } 10)$

$x = 4 \Rightarrow x_1 = 4$

$x = 9 \Rightarrow x_2 = 9$

$9x_2 = 1 \ (\text{in } Z_{10})$

$x_2 = 9$

$\therefore X_1 = (10 \times 2 \times 1 + 9 \times 4 \times 9) \text{ mod } 90$

$= 37$

$X_2 = (10 \times 2 \times 1 + 9 \times 9 \times 9) \text{ mod } 90$

$= 29$

2

$x \ (\text{mod } 9)$

2

QUESTION 15. (5 points)

(1) Find all possible solution of $12x = 16$ over PLANET Z_{20}

$\gcd(12, 20) = 4$

$\therefore$ There are 4 solutions.

$x = \{3, 8, 13, 18\}$

$20 = 4 \times \boxed{5}$

$20 \overline{)36}$, 1, 20, 16

$20 \overline{)96}$, 4, 80, 16

$20 \overline{)156}$, 7, 140

(2) Find all possible solution of $12x \ (mod\ 20) = 16$ over PLANET Z

~~gcd~~

$x_1 = 3 + 5k \quad (k \in Z)$

Faculty information

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